

# LARGE DEVIATIONS BOUND FOR LORENZ-LIKE ATTRACTORS

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**ABSTRACT.** We obtain an exponential large deviation upper bound for continuous observables on suspension semiflows over a non-uniformly expanding base transformation with non-flat singularities (or criticalities), where the roof function defining the suspension behaves like the logarithm of the distance to the singular/critical set of the base map. Suspension semiflows model the dynamics of flows admitting cross-sections, where the dynamics of the base is given by the Poincaré return map and the roof function is the return time to the cross-section. The results are applicable in particular to semiflows modeling the geometric Lorenz attractors and the Lorenz flow.

## 1. INTRODUCTION

Given a Hölder- $C^1$  local diffeomorphism  $f : M \setminus \mathcal{S} \rightarrow M$  outside a volume zero non-flat<sup>1</sup> singular set  $\mathcal{S}$ , let  $X^t : M_r \rightarrow M_r$  be a *semiflow with roof function*  $r : M \setminus \mathcal{S} \rightarrow \mathbb{R}$  over the base transformation  $f$ , as follows.

Set  $M_r = \{(x, y) \in M \times [0, +\infty) : 0 \leq y < r(x)\}$  and  $X^0$  the identity on  $M_r$ , where  $M$  is a compact Riemannian manifold. For  $x = x_0 \in M$  denote by  $x_n$  the  $n$ th iterate  $f^n(x_0)$  for  $n \geq 0$ . Denote  $S_n^f \varphi(x_0) = \sum_{j=0}^{n-1} \varphi(x_j)$  for  $n \geq 1$  and for any given real function  $\varphi$ . Then for each pair  $(x_0, s_0) \in X_r$  and  $t > 0$  there exists a unique  $n \geq 1$  such that  $S_n r(x_0) \leq s_0 + t < S_{n+1} r(x_0)$  and define

$$X^t(x_0, s_0) = (x_n, s_0 + t - S_n r(x_0)).$$

The study of suspension (or special) flows is motivated by modeling a flow admitting a cross-section. Such flow is equivalent to a suspension semiflow over the Poincaré return map to the cross-section with roof function given by the return time function on the cross-section. This is a main tool in the ergodic theory of uniformly hyperbolic flows developed by Bowen and Ruelle[6].

An invariant probability measure  $\mu$  for a flow  $X^t$  on a compact manifold is a *physical (or Sinai-Ruelle-Bowen) probability measure* if the points  $z$  satisfying for all continuous functions  $\psi$

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \psi(X^s(z)) ds = \int \psi d\mu,$$

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<sup>1</sup> $f$  behaves like a power of the distance to  $\mathcal{S}$ :  $\|Df(x)\| \approx \text{dist}(x, \mathcal{S})^{-\beta}$  for some  $\beta > 0$  (see Alves-Araújo[1] for a precise statement).

form a subset with positive volume<sup>2</sup> on the ambient space. These measures are an object of intense study[5].

Having such invariant probability measures it is natural to consider the rate of convergence of the time averages to the space average, measured by the volume of the subset of points whose time averages stay away from the space average by a prescribed amount up to some evolution time. We extend part of the results on large deviation rates of Kifer [7] from the uniformly hyperbolic setting to semiflows over non-uniformly expanding base dynamics and unbounded roof function. These special flows model non-uniformly hyperbolic flows like the Lorenz flow, exhibiting equilibria accumulated by regular orbits.

**1.1. Conditions on the base dynamics.** We assume that the singular set  $\mathcal{S}$  (containing the points where  $f$  is either *not defined*, *discontinuous* or *not differentiable*) is regular, e.g. a submanifold of  $M$ , and that  $f$  is *non-uniformly expanding*: there exists  $c > 0$  such that for Lebesgue almost every  $x \in M$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} S_n \psi(x) \leq -c \quad \text{where} \quad \psi(x) = \log \|Df(x)^{-1}\|.$$

Moreover we assume that  $f$  has *exponentially slow recurrence to the singular set*  $\mathcal{S}$  i.e. for all  $\epsilon > 0$  there is  $\delta > 0$  s.t.

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \text{Leb} \left\{ x \in M : \frac{1}{n} S_n |\log d_\delta(x, \mathcal{S})| > \epsilon \right\} < 0,$$

where  $d_\delta(x, y) = d(x, y)$  if  $d(x, y) < \delta$  and  $d_\delta(x, y) = 1$  otherwise<sup>3</sup>.

These conditions ensure[2] in particular the existence of finitely many ergodic absolutely continuous (in particular *physical*)  $f$ -invariant probability measures  $\mu_1, \dots, \mu_k$  whose basins cover the manifold Lebesgue almost everywhere.

We say that an  $f$ -invariant measure  $\mu$  is an *equilibrium state* with respect to the potential  $\log J$ , where  $J = |\det Df|$ , if  $h_\mu(f) = \mu(\log J)$ , that is if  $\mu$  *satisfies the Entropy Formula*. Denote by  $\mathbb{E}$  the family of all such equilibrium states. It is not difficult to see that each physical measure in our setting belongs to  $\mathbb{E}$ .

We assume that  $\mathbb{E}$  is *formed by a unique absolutely continuous probability measure*.

**1.2. Conditions on the roof function.** We assume that  $r : M \setminus \mathcal{S} \rightarrow \mathbb{R}^+$  has *logarithmic growth near  $\mathcal{S}$* : there exists  $K = K(\varphi) > 0$  such that<sup>4</sup>  $r \cdot \chi_{B(\mathcal{S}, \delta)} \leq K \cdot |\log d_\delta(x, \mathcal{S})|$  for all small enough  $\delta > 0$ . We also assume that  $r$  is bounded from below by some  $r_0 > 0$ .

**1.3. Main result.**

**Theorem A.** *Let  $X^t$  be a suspension semiflow over a non-uniformly expanding transformation  $f$  on the base  $M$ , with roof function  $r$ , satisfying all the previously stated conditions.*

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<sup>2</sup>given by a normalized volume form induced by the Riemannian metric on  $M$ , also referred to as Lebesgue measure

<sup>3</sup> $d$  here is the Riemannian distance on  $M$ .

<sup>4</sup> $B(\mathcal{S}, \delta)$  is the  $\delta$ -neighborhood of  $\mathcal{S}$ .

Let  $\psi : M_r \rightarrow \mathbb{R}$  be continuous,  $\nu = \mu \times \text{Leb}^1$  be the induced invariant measure<sup>5</sup> for the semiflow  $X^t$  and  $\lambda = \text{Leb} \times \text{Leb}^1$  be the natural extension of volume to the space  $M_r$ . Then

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \lambda \left\{ z \in M_r : \left| \frac{1}{T} \int_0^T \psi(X^t(z)) dt - \nu(\psi) \right| > \epsilon \right\} < 0.$$

**1.4. Consequences for the Lorenz flow.** The Lorenz equations

$$\dot{x} = 10(y - x), \quad \dot{y} = 28x - y - xz, \quad \dot{z} = xy - 8z/3$$

were presented by Lorenz[8] as a simplified model of convection on the Earth's atmosphere. Recently Tucker[9] showed that the above equations and similar equations with nearby parameters define a geometric Lorenz flow: we may find a **global cross-section** given by a (embedded) square whose Poincaré first return map provides a *roof function with logarithmic growth near the singularity line* and having a *uniformly contracting one-dimensional foliation*.

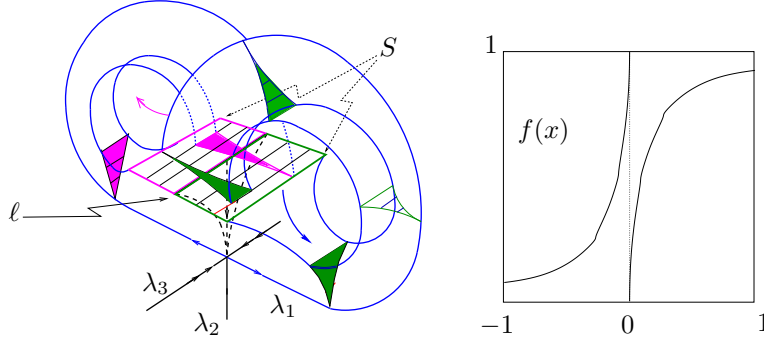


FIGURE 1. The global cross-section for the geometric Lorenz flow and the associated 1d quotient map, the Lorenz transformation.

The one-dimensional map  $f$  obtained quotienting over the leaves of the stable foliation has all the properties stated previously for the base transformation. The Poincaré return time gives also a roof function with logarithmic growth near the singularity line.

The uniform contraction along the stable leaves implies that the *time averages of two orbits on the same stable leaf under the first return map are uniformly close* for all big enough iterates. If  $P : S \rightarrow [-1, 1]$  is the projection along stable leaves

**Lemma 1.1.** *For  $\varphi : U \supset \Lambda \rightarrow \mathbb{R}$  continuous and bounded,  $\epsilon > 0$  and  $\varphi(x) = \int_0^{r(x)} \psi(x, t) dt$ , there exists  $\zeta : [-1, 1] \setminus \mathcal{S} \rightarrow \mathbb{R}$  with logarithmic growth near  $\mathcal{S}$  such that  $\left\{ \left| \frac{1}{n} S_n^R \varphi - \mu(\varphi) \right| > 2\epsilon \right\} \subseteq P^{-1} \left( \left\{ \left| \frac{1}{n} S_n^f \zeta - \mu(\zeta) \right| > \epsilon \right\} \cup \left\{ \left| \frac{1}{n} S_n^f \right| \log \text{dist}_\delta(y, \mathcal{S}) \right| > \epsilon \right\} \right)$*

Hence in this setting it is enough to study the quotient map  $f$  to get information about deviations for the Poincaré return map. Coupled with the main result we are then able to deduce

<sup>5</sup>for any  $A \subset M_r$  set  $\nu(A) = \mu(r)^{-1} \int d\mu(x) \int_0^{r(x)} ds \chi_A(x, s)$ .

**Corollary B.** *Let  $X^t$  be a flow on  $\mathbb{R}^3$  exhibiting a Lorenz or a geometric Lorenz attractor with trapping region  $U$ . Denoting by  $\text{Leb}$  the normalized restriction of the Lebesgue volume measure to  $U$ ,  $\psi : U \rightarrow \mathbb{R}$  a bounded continuous function and  $\mu$  the unique physical measure for the attractor, then for any given  $\epsilon > 0$*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \text{Leb} \left\{ z \in U : \left| \frac{1}{T} \int_0^T \psi(X^t(z)) dt - \mu(\psi) \right| > \epsilon \right\} < 0.$$

Moreover for any compact  $K \subset U$  such that  $\mu(K) < 1$  we have

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \log \text{Leb} \left( \{x \in K : X^t(x) \in K, 0 < t < T\} \right) < 0.$$

## 2. IDEA OF THE PROOF

We use properties of non-uniformly expanding transformations, especially a large deviation bound recently obtained[3], to deduce a large deviation bound for the suspension semiflow reducing the estimate of the volume of the deviation set to the volume of a certain deviation set for the base transformation.

The initial step of the reduction is as follows. For a continuous and bounded  $\psi : M_r \rightarrow \mathbb{R}$ ,  $T > 0$  and  $z = (x, s)$  with  $x \in M$  and  $0 \leq s < r(x) < \infty$ , there exists the **lap number**  $n = n(x, s, T) \in \mathbb{N}$  such that  $S_n r(x) \leq s + T < S_{n+1} r(x)$ , and we can write

$$\begin{aligned} \int_0^T \psi(X^t(z)) dt &= \int_s^{r(x)} \psi(X^t(x, 0)) dt + \int_0^{T+s-S_n r(x)} \psi(X^t(f^n(x), 0)) dt \\ &\quad + \sum_{j=1}^{n-1} \int_0^{r(f^j(x))} \psi(X^t(f^j(x), 0)) dt. \end{aligned}$$

Setting  $\varphi(x) = \int_0^{r(x)} \psi(x, 0) dt$  we can rewrite the last summation above as  $S_n \varphi(x)$ . We get the following expression for the time average

$$\begin{aligned} \frac{1}{T} \int_0^T \psi(X^t(z)) dt &= \frac{1}{T} S_n \varphi(x) - \frac{1}{T} \int_0^s \psi(X^t(x, 0)) dt \\ &\quad + \frac{1}{T} \int_0^{T+s-S_n r(x)} \psi(X^t(f^n(x), 0)) dt. \end{aligned}$$

Writing  $I = I(x, s, T)$  for the sum of the last two integral terms above, observe that for  $\omega > 0$ ,  $0 \leq s < r(x)$  and  $n = n(x, s, T)$

$$\left\{ (x, s) \in M_r : \left| \frac{1}{T} S_n \varphi(x) + I(x, s, T) - \frac{\mu(\varphi)}{\mu(r)} \right| > \omega \right\}$$

is contained in

$$\left\{ (x, s) \in M_r : \left| \frac{1}{T} S_n \varphi(x) - \frac{\mu(\varphi)}{\mu(r)} \right| > \frac{\omega}{2} \right\} \cup \left\{ (x, s) \in M_r : I(x, s, T) > \frac{\omega}{2} \right\}.$$

The left hand side above is a *deviation set for the observable  $\varphi$  over the base transformation*, while the right hand side will be *bounded by the geometric conditions on  $S$  and by a deviations bound for the observable  $r$  over the base transformation*.

Analysing each set using the conditions on  $f$  and  $r$  and noting that for  $\mu$ - and Leb-almost every  $x \in M$  and every  $0 \leq s < r(x)$

$$\frac{S_n r(x)}{n} \leq \frac{T+s}{n} \leq \frac{S_{n+1} r(x)}{n} \quad \text{so} \quad \frac{n(x, s, T)}{T} \xrightarrow{T \rightarrow \infty} \frac{1}{\mu(r)},$$

we are able to obtain the asymptotic bound of the Main Theorem.

Full details of the proof are presented in [4].

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